Graduate Classical Mechanics

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Acknowledgements

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Contents

Chapter 1

Newtonian Mechanics

"If I have seen further it is by standing on ye shoulders of Giants." - Sir Isaac Newton (1676)

When Sir Isaac Newton published the *Principia* in 1686, he laid out the basis of all physics from particles colliding in an accelerator to planets orbiting about one another. Granted, there are more nuances to these systems (quantum mechanics and general relativity respectively), but to understand these phenomena, we build on the foundation laid out by Newton.

Before the discovery of quantum mechanics and general relativity, it was thought that physics was done, that we had discovered everything and the only thing remaining was more precise measurements. This statement was proven false, and it was found that classical mechanics does not provide a perfect interpretation of the world around us. That being said, why study classical mechanics if it is not entirely correct? I can think of two reasons (not conclusive). First, classical mechanics provides a good approximation for most situations. Quantum mechanics really only kick in around the Planck scale. For anything bigger, we can use classical mechanics, and it gives us the answer to a rather high degree of precision. Relativistic corrections are needed as we approach the speed of light, but the fastest man-made object (a nuclear powered steel plate) only reached upwards of 66 kilometers per hour, on the order of ten-thousand times less than the speed of light. Most things we deal with are on classical scales. It would be rather tedious and pointless if we had to make relativistic corrections every time we tried to do a block-sliding-down-a-ramp problem. Second, classical mechanics, as stated before, is the starting point of physics. Much of the math and language we use later is developed here.

1.1 Newton's Laws

We'll start with the very basics: Newton's three laws. If you've taken any physics class, you will almost certainly have had these drilled into your head. In this section, we'll restate them and set up some notation.

1.1.1 First Law

Imagine we have a particle, which in classical mechanics need not be subatomic (and indeed should not be subatomic. Note that we will run into problems involving subatomic particles, we just treat them classically). It could be anything from a billiard ball to a planet. In any case, we can specify its position at some time by a vector $\vec{x}(t)$. If we ask that particle where it came from and where it's going, we need to know its velocity, $\vec{v}(t)$ (1.1). The dot over a variable means we take the time-derivative. When Newton wrote the Principia, he dealt with momentum, defined to be the product of a particle's mass and velocity (1.2). In your undergraduate classes, mass was generally kept constant, but now that we're at the graduate level, we can see what happens when the mass is not constant. Finally, we can ask what happens if the velocity changes. This gives us the acceleration of the system (1.3), which as we shall see shortly, is tied to the force. There are higher order terms (jerk, snap, crackle, pop, lock, and drop if I remember correctly), but we will largely ignore these terms for now.

$$\vec{v}(t) = \dot{\vec{x}} = \frac{d\vec{x}}{dt} \tag{1.1}$$

$$\vec{p}(t) = m(t)\ \vec{v}(t) \tag{1.2}$$

$$\vec{a}(t) = \ddot{\vec{x}} = \frac{d^2\vec{x}}{dt^2} \tag{1.3}$$

If we have a particle moving not in a straight line, it is sometimes easier to work in angular coordinates. For velocity and acceleration, they follow the same formulas laid out above, but they are commonly labeled as $\vec{\omega}(t)$ and $\vec{\alpha}(t)$ respectively. Position is often denoted by $\vec{r}(t)$ or $\vec{\theta}(t)$. N.B., I will likely use $\vec{r}$ and $\vec{x}$ interchangeably. The main difference from linear coordinates is angular momentum (1.4). Note that compared to linear momentum (1.2), we need to specify a point of origin.

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t) \tag{1.4}$$

Now that we have some notation out of the way, we can get to stating Newton's First Law. Put simply, Newton's First Law states that an object moving with a constant velocity will continue to travel at that speed unless acted upon by some outside force. This seems intuitive to us today, but when first stated, it flew in the face of thousands of years of science (referred to as Aristotelian thought). To put this mathematically, imagine that we have some particle. If we know its velocity and position at some time t_0, then the position a time Δt later is given by

$$\vec{x}(t_0 + \Delta t) = \vec{x}(t_0) + \vec{v}(t_0)\Delta t$$

However, this isn't actually true for most reference frames. For example, imagine that our reference frame is rotating. In this case, the particle will not travel at a constant velocity, i.e., $\ddot{\vec{x}} \neq 0$. In order for Newton's First Law to hold true, we have to be in an inertial reference frame, one in which particles travel at constant velocity when the forces acting on them vanish. We can restate Newton's First Law as *inertial frames exist*. From here on out, we will work almost entirely in inertial frames (this statement might turn out to be horrendously false).

One thing to note is that inertial frames are invariant under translation, rotation, and boosting. That is, if we move our frame somewhere else, rotate it, or move it at a constant velocity, physics will act the same as if we had done nothing to the frame. This is referred to as the principle of relativity. There is no special origin, direction, or velocity in the universe (well, the speed of light is special, but for other reasons).

One way to think about the principle of relativity is by picturing an elevator. If you are isolated inside that elevator, it is impossible to tell if it is at rest or moving at a constant velocity. The only way to tell if the elevator is moving is if it is speeding up or slowing down. An accelerating frame is not an inertial frame, so physics will behave differently.

1.1.2 Second Law

Newton's Second Law is probably his best known. It is a concise way to write the equations of motion for a particle. We define some quantity force $\vec{F}$ as the change in momentum (1.5). The angular analog is torque (1.6). We can convince ourselves that for a particle with constant mass, equation (1.5) turns into the familiar $F = ma$.

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \tag{1.5}$$

$$\vec{N} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} = \dot{\vec{L}} \tag{1.6}$$

One thing we should note is that when we start looking at the forces between particles, we have to introduce the concept of a field. In Newtonian thinking, forces are created and destroyed instantaneously. However, this doesn't quite sit well with Einstein's claim that the speed of light is the ultimate speed limit in the universe. Imagine if the sun were to disappear. According to Newton, Earth would then fly off in a tangential direction. The gravitational force from the sun would be travelling faster than the speed of light, but that can't happens. So how do we reconcile this? We imagine there is some underlying sheet (or field) that manifests these forces. Eventually, we'll see that things like particles are actually the product of quantizing these fields, but we'll burn that bridge when we get to it.

1.1.3 Third Law

Newton's Third Law states that if there are two particles, the force of one particle acting on the other is equal in magnitude and opposite in direction to the force from the other particle, $\vec{F}_{ij} = -\vec{F}_{ji}$. We can imagine that if this weren't true and we pushed our hands together, they would actually go through each other since the acceleration of one hand would be greater than the acceleration of the other.

The other consequence of Newton's Third Law is that if there are no external forces in a system, the net force is zero. Looking at equation (1.5), if $\vec{F} = 0$, the momentum of the system does not change. Momentum is a conserved quantity. We can also convince ourselves that angular momentum is a conserved quantity, i.e., if there are no external torques, angular momentum does not change.

There are several conserved quantities in physics, but the main one we deal with in classical mechanics is the conservation of energy. If we move a particle from point 1 to point 2, we do some work on that particle (1.7). That work translates to a change in energy, which is referred to as the Work Energy Theorem (1.8).

If you've ever been on a roller-coaster, you know that the cart comes to a stop when it reaches the top of that first hill only to pick up speed as it goes down until it is moving quite fast. At the start, the cart has all potential energy (1.10) (it has the potential to move), and at the end, that potential energy has changed into kinetic energy (1.9). If there were no outside forces, the potential energy at the top of the hill is equal to the kinetic energy at the bottom of the hill.

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} \tag{1.7}$$

$$W_{12} = T_2 - T_1 \tag{1.8}$$

$$T = {}^1\!/\!_2\, mv^2 \tag{1.9}$$

$$\vec{F} = -\nabla V(\vec{r}) \tag{1.10}$$

One thing to note is that the work energy theorem (1.8) only holds for conservative forces (1.11). Normally, we would have to worry about the path, but for conservative forces, we only care about the end points. Another way to say this is to think about a particle being brought from point 1 to point 2 along one path then brought back to point 1 along a different path. If the total work done on the system is 0, the force is conservative.

$$\oint \vec{F} \cdot \, d\vec{s} = 0 \tag{1.11}$$

Further, it can be shown that if the forces acting on a particle are conservative, the total energy of the particle is also conserved and can be written as the sum of the kinetic and the potential energy at any point.

$$\frac{1}{2} m\dot{x}^2 + V(x) = E$$

We can isolate the velocity term,

$$\frac{dx}{dt} = \pm\sqrt{{}^2\!/\!_m(E - V(x))}$$

$$dt = \pm\frac{dx}{\sqrt{{}^2\!/\!_m(E - V(x))}}$$

Integrating both sides, we get a solution for the equations of motion referred to as quadrature (1.12). Normally, this integral cannot be solved, so solutions will sometimes be referred to as having been "reduced to quadrature".

$$t - t_0 = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - V(x')}} \tag{1.12}$$

You should now be able to do Goldstein 1.1, 1.11, 1.12, and 1.13.

You should now be able to do Jose 1.1, 1.3, 1.7, 1.9, 1.10, 1.11, 1.13, 1.14, 1.15, and 1.20.

1.2 Mechanics of Multiple Particle Systems

In the previous section, we looked at a system consisting of a single particle. Naturally, the next step is to have multiple particles. The first thing we do is label each particle with some subscript i. So the position of particle i is x_i, and its mass is m_i. Thankfully, other than that, not much else changes, Newton's Laws don't morph into some radical new form.

1.2.1 Newton's Second Law

That being said, we should mention that Newton's Second Law (1.5) does change a little. We can break up the force a particle feels into either external or internal forces. External forces are, as the name suggests, forces from outside the system. Internal forces are forces from other particles in the system. If we sum over all particles, we find that only the external force remains since the internal forces cancel out by Newton's Third Law. We can write Newton's Second Law on particle i in the form of equation (1.13) where $\vec{F}_i^{(e)}$ are the external forces acting on particle i.

$$\sum_{j\neq i} \vec{F}_{ij} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i \tag{1.13}$$

1.2.2 Center of Mass

We start by introducing a concept central to multiple particle systems. If we have a system consisting of N particles, the total mass of that system can be found by summing up the mass of each particle,

$$M = \sum_{i=1}^{N} m_i$$

Similarly, imagine we have a continuous mass distribution (which we will write as $\rho(x)$). To find the total mass, we integrate over all space. For a single-dimensional system, this becomes

$$M = \int \rho(x)\ dx$$

See my text on Classical Electrodynamics for how we can write discrete systems using the continuous formulation by strategically using delta functions.

Averaging the distances and masses of each particle, we can find the center of mass (1.14). If we have a continuous system, we use equation (1.15).

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i \tag{1.14}$$

$$\vec{R} = \frac{1}{M} \int \rho(\vec{r})\vec{r}\ dV \tag{1.15}$$

One important concept that jumps out of the center of mass is that we don't really need to care about the individual particles in a system, we just need to know what happens to the center of mass. You may have seen a variation on this problem in your undergraduate class. Imagine you have a ball flying through the air. Suddenly, it breaks apart in mid-flight. The individual pieces will fly off in different directions, but the center of mass will follow the trajectory the ball would have followed had it remained whole. We can see this if we look at the force on the center of mass,

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2}$$

Alternatively, the total force on the system can be found by summing over the force on each particle (1.13),

$$\frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_{i\neq j}\vec{F}_{ji}$$

However, remember that Newton's Third Law says that $F_{ij} = -F_{ji}$, so when we sum over all i and j, that last term cancels so we're left with

$$\vec{F} = \vec{F}^{(e)}$$

For a system of particles, we only need to look at the external forces, we don't care about the internal forces. We can do something similar for the momenta. We can rewrite the total momentum of a system (1.16) in terms of the center of mass.

$$\vec{P} = \sum_{i=1}^{N} m_i\dot{\vec{r}}_i$$

$$\vec{P} = M\frac{d\vec{R}}{dt} \tag{1.16}$$

For angular momentum, we find the total (1.17) by taking the cross product $\vec{r}_i \times \vec{p}_i$ and summing over all i. This is a little more difficult to derive since we have to worry about our reference point. We start by defining the position and velocity from the origin to particle i,

$$\begin{cases} \vec{r}_i = \vec{r}_i' + \vec{R} \\ \vec{v}_i = \vec{v}_i' + \vec{v} \end{cases}$$

where $\vec{r}_i$ and $\vec{v}_i$ are the position and velocity of particle i relative to the center of mass. Substituting these in,

$$\vec{L} = \sum_i(\vec{r}_i' + \vec{R}) \times m_i(\vec{v}_i' + \vec{v})$$

$$= \sum_i \vec{r}_i' \times m_i\vec{v}_i' + \sum_i \vec{r}_i' \times m_i\vec{v} + \sum_i \vec{R} \times m_i\vec{v}_i' + \sum_i \vec{R} \times m_i\vec{v}$$

$$= \sum_i \vec{R} \times m_i\vec{v} + \sum_i \vec{r}_i' \times m_i\vec{v}_i' + \left(\sum_i m_i\vec{r}_i'\right) \times \vec{v} + \vec{R} \times \frac{d}{dt}\sum_i m_i\vec{r}_i'$$

The last two terms disappear since we sum over $m_i \vec{r}_i'$, which is the vector from the center of mass to the particle. Summing over all of these vectors, they cancel out, and we are left with the total angular momentum (1.17). The first term is the angular momentum of the center of mass, and the second term is the angular momentum about the center of mass.

$$\vec{L} = \vec{R} \times M\vec{v} + \sum_i \vec{r}_i' \times \vec{p}_i' \tag{1.17}$$

1.2.3 Collisions and Energy

There are two types of collisions that we will deal with. The first is elastic collisions. An example of an elastic collision would be two pool balls bouncing off of each other. Here, both energy and momentum are conserved. The other type of collision we will deal with is inelastic collision. An example would be a hockey puck colliding with an octopus and both of them sliding together. Kinetic energy is not conserved, but momentum is. We can convince ourselves that energy is not conserved in this system by thinking about the system in reverse. Imagine we have an object moving along, and it spontaneously splits into two pieces. In order to do this, we need to introduce some energy into the system.

Speaking of energy, how does energy change when we have multiple particles? For kinetic energy, we sum the kinetic energy of each particle (1.9),

$$T = {}^1\!/_2 \sum_i m_i \vec{v}_i \cdot \vec{v}_i$$

Using the notation in the previous section for a particle about the center of mass,

$$= {}^1\!/_2 \sum_i m_i (\vec{v} + \vec{v}_i') \cdot (\vec{v} + \vec{v}_i')$$

$$= {}^1\!/_2 \sum_i m_i v^2 + \sum_i m_i \vec{v} \cdot \vec{v}_i' + {}^1\!/_2 \sum_i m_i v_i'^2$$

$$= {}^1\!/_2 \sum_i m_1 v^2 + {}^1\!/_2 \sum_i m_i v_i'^2 + \vec{v} \cdot \frac{d}{dt} \left(\sum_i m_i \vec{r}_i' \right)$$

Again, the last term disappears, and we are left with the total kinetic energy (1.18). The first term is the kinetic energy of the center of mass, and the second term is the kinetic energy of the particles in relation to the center of mass.

$$T = {}^1\!/_2 M v^2 + {}^1\!/_2 \sum_i m_i v_i'^2 \tag{1.18}$$

The potential energy is broken up into external forces (1.19) and internal forces (1.20).

$$\vec{F}_i^{(e)} = -\nabla_i V_i(\vec{x}_i) \tag{1.19}$$

$$F_{ij} = -\nabla_i V_{ij}(|\vec{x}_i - \vec{x}_j|) \tag{1.20}$$

Summing all of these up, we get the total conserved energy (1.21).

$$E = T + \sum_i V_i(\vec{x}_i) + \sum_{i<j} V_{ij}(|\vec{x}_i - \vec{x}_j|) \tag{1.21}$$

1.2.4 Example: Turning a Two Body Problem into a One Body Problem

Imagine we have two particles with different masses and different positions. If there are no external forces, we can reduce this to a one-particle problem. First, we know the center of mass (1.14),

$$M\vec{R} = m_1\vec{x}_1 + m_2\vec{x}_2$$

We also want to define the relative separation,

$$\vec{r} = \vec{x}_1 - \vec{x}_2$$

We can then write the position of each particle using the center of mass and relative separation,

$$\begin{cases} \vec{x}_1 = \vec{R} + \dfrac{m_2}{M}\vec{r} \\ \\ \vec{x}_2 = \vec{R} - \dfrac{m_1}{M}\vec{r} \end{cases}$$

Since there are no external forces, the center of mass has no force acting on it. The relative motion of the particles however,

$$\ddot{\vec{r}} = \ddot{\vec{x}}_1 - \ddot{\vec{x}}_2$$

$$= \frac{1}{m_1}\vec{F}_{12} - \frac{1}{m_2}\vec{F}_{21}$$

Using Newton's Third Law,

$$= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}$$

$$\ddot{\vec{r}} = \frac{m_1 + m_2}{m_1 m_2}\vec{F}_{12}$$

We could also write this as

$$\mu\ddot{\vec{r}} = \vec{F}_{12}$$

where μ is the reduced mass (1.22), and we've reduced this two particle problem to a one-particle problem.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.22}$$

You should now be able to do Goldstein 1.2, 1.3, 1.14, 1.17, and 1.23.

You should now be able to do Jose 1.2, 1.4, 1.5, 1.6, 1.17, and 1.18.

1.3 Constraints

A constraint is some condition that we need to take into account that restricts the motion of a particle. For example, we can use the equations of motion to figure out what happens when we drop a ball. However, we are at a bit of a loss if we then introduce a floor that stops the ball from falling. We need to look at how to deal with constraints in order to solve this problem.

1.3.1 Generalized Coordinates

Imagine we have a system consisting of N particles. If there are no constraints, that system will have $3N$ independent coordinates or degrees of freedom. That is, to fully describe the system, we need to use $3N$ variables (three spatial coordinates for each particle). As we shall see, constraints help to reduce the number of degrees of freedom. If we have k constraint equations, then we have $3N - k$ degrees of freedom.

1.3.2 Constraint Equations

There are two types of constraint equations we will deal with: holonomic and non-holonomic. If the constraint can be expressed using equation (1.23). Holonomic constraints depend only on position and time, not on any higher order time derivatives. If a constraint cannot be reduced to equation (1.23), it is non-holonomic.

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., t) = 0 \tag{1.23}$$

For example, a bead constrained to move on a string is holonomic. If that string is the x-axis, we would write the constraint as

$$x = 0$$

which is of the form (1.23). On the other hand, if we place a particle on the surface of a sphere, that is a non-holonomic constraint since we would write it as

$$r^2 - a^2 \geq 0$$

The inequality makes it non-holonomic.

Another classification we could use is rheonomous and scleronomous constraints. Rheonomous constraints explicitly depend on time while scleronomous do not. Both of the examples above are scleronomous. However, if we have that string move up and down, it would become a rheonomous constraint.

One common constraint we should mention is rolling without slipping. If a disk rolls without slipping, the distance it covers is equal to the amount it rotates. (1.24). That is, if we have disk of radius 1, if it makes a full rotation without slipping, it will have moved a linear distance of 2π.

$$v = a\dot{\phi} \tag{1.24}$$

You should now be able to do Goldstein 1.4, 1.5, and 1.6.

You should now be able to do Jose 1.12 and 1.26.