Solutions to Classical Dynamics: A Contemporary Approach by Jose and Saletan

Benjamin D. Suh

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Chapter 1

Fundamentals of Mechanics

1.1 Parabolic Motion

A gun is mounted on a hill of height h above a level plane. Neglecting air resistance, find the angle of elevation α for the greatest horizontal range at a given muzzle speed v. Find this range.

Since there is no horizontal force, the distance traveled horizontally is

$$d = vt\cos(\alpha)$$

In the vertical direction, we have gravitational force,

$$-h = vt\sin(\alpha) - {}^{1}\!/_{2}\, gt^2$$

Taking the vertical equation, we can use the quadratic formula to solve for t. We also want to keep the positive value,

$$t = \frac{v\sin(\alpha) + \sqrt{v^2\sin^2(\alpha) + 2gh}}{g}$$

Substituting in distance,

$$\frac{d}{v\cos(\alpha)} = \frac{v\sin(\alpha) + \sqrt{v^2\sin^2(\alpha) + 2gh}}{g}$$

$$d = \frac{v^2\sin(\alpha)\cos(\alpha) + v\cos(\alpha)\sqrt{v^2\sin^2(\alpha) + 2gh}}{g}$$

We want to maximize this value, so we take the derivative according to α and set the result equal to 0. First, we get rid some of constants by dividing by v^2/g,

$$\frac{\partial d}{\partial \alpha} = \cos^2(\alpha) - \sin^2(\alpha) - \sin(\alpha)\sqrt{\sin^2(\alpha) + \frac{2gh}{v^2}} + \sin(\alpha)\cos^2(\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{-1/2} = 0$$

Using the relation $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and multiplying through by the inverse root part,

$$\cos(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{1/2} - \sin(\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right) + \sin(\alpha)\cos^2(\alpha) = 0$$

$$\cos(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{1/2} = \sin(\alpha)\left(\sin^2(\alpha) - \cos^2(\alpha) + \frac{2gh}{v^2}\right)$$

$$\cos(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{1/2} = \sin(\alpha)\left(\frac{2gh}{v^2} - \cos(2\alpha)\right)$$

$$\cos^2(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right) = \sin^2(\alpha)\left(\left(\frac{2gh}{v^2}\right)^2 - \frac{4gh}{v^2}\cos(2\alpha) + \cos^2(2\alpha)\right)$$

$$\frac{2gh}{v^2}\left(\cos^2(2\alpha) + 2\cos(2\alpha)\sin^2(\alpha)\right) = \left(\frac{2gh}{v^2}\right)^2 \sin^2(\alpha)$$

Using the relation $\sin^2(\alpha) = 1/2\,(1 - \cos(2\alpha))$,

$$\cos(2\alpha) = \frac{1}{2}\left(\frac{2gh}{v^2}\right)(1 - \cos(2\alpha))$$

$$\cos(2\alpha)\left(1 + \frac{gh}{v^2}\right) = \frac{gh}{v^2}$$

$$\cos(2\alpha) = \frac{gh}{v^2 + gh}$$

To find the maximum range, we plug this back into our equation for the distance. It is easiest if we convert all of the trigonometric functions to $\cos(2\alpha)$,

$$\begin{cases} \cos(\alpha) = \sqrt{\dfrac{1 + \cos(2\alpha)}{2}} = \sqrt{\dfrac{v^2 + 2gh}{2(v^2 + gh)}} \\[2ex] \sin(\alpha) = \sqrt{\dfrac{1 - \cos(2\alpha)}{2}} = \sqrt{\dfrac{v^2}{2(v^2 + gh)}} \end{cases}$$

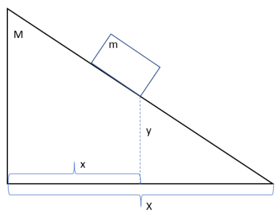
$$R = \frac{1}{g}\left(v^2\frac{\sqrt{v^2(v^2+2gh)}}{2(v^2+gh)} + v\sqrt{\frac{v^2+2gh}{2(v^2+gh)}}\sqrt{\frac{v^4}{2(v^2+gh)}+2gh}\right)$$

$$= \frac{1}{g}\left(\frac{v^3\sqrt{v^2+2gh}}{2(v^2+gh)} + \frac{v(v^2+2gh)\sqrt{v^2+2gh}}{2(v^2+gh)}\right)$$

$$R = \frac{v}{g}\sqrt{v^2+2gh}$$

1.2 Block on a Ramp

A mass m slides without friction on a plane tilted at an angle θ in a vertical uniform gravitational field g. The plane itself is on rollers and is free to move horizontally, also without friction; it has mass M. Find the acceleration A of the plane and the acceleration a of the mass m.



We draw the block as in figure (1.2) with X being the position of the block. From the diagram, we get a relation between the positions and the angles,

$$\tan(\theta) = \frac{y}{X - x}$$

$$X = x + y\cot(\theta)$$

Taking the time derivative twice,

$$\ddot{X} = \ddot{x} + \ddot{y}\cot(\theta)$$

Since there are no external forces in the x-direction, we can use conservation of momentum. Since everything is initially at rest,

$$0 = M\dot{X} + m\dot{x}$$

We now want to take a time derivative,

$$M\ddot{X} + m\ddot{x} = 0$$

From drawing free body diagrams, we can get the forces on each component,

$$\begin{cases} m\ddot{x} = N\sin(\theta) \\ m\ddot{y} = -mg + N\cos(\theta) \end{cases}$$

$$\begin{cases} M\ddot{X} = -N\sin(\theta) \\ M\ddot{Y} = 0 \end{cases}$$

We can now solve for the necessary values. We'll start by relating X and x,

$$\ddot{X} = -\frac{m}{M}\ddot{x}$$

$$-\ddot{y}\cot(\theta) = \left(1 + \frac{m}{M}\right)\ddot{x}$$

Alternatively,

$$m\ddot{y} = -mg + m\ddot{x}\cot(\theta)$$

Combining these two,

$$\ddot{y} = -g - \left(\frac{M}{m+M}\right)\ddot{y}\cot^2(\theta)$$

$$\ddot{y}\left(1 + \frac{M\cos^2(\theta)}{(M+m)\sin^2(\theta)}\right) = -g$$

$$\ddot{y} = -g\left(\frac{(M+m)\sin^2(\theta)}{M+m\sin^2(\theta)}\right)$$

$$\ddot{x} = -\ddot{y}\left(\frac{M}{m+M}\right)\cot(\theta)$$

$$= g\frac{\cos(\theta)}{\sin(\theta)}\left(\frac{M}{m+M}\right)\left(\frac{(m+M)\sin^2(\theta)}{m\sin^2(\theta)+M}\right)$$

$$\ddot{x} = g\left(\frac{M\sin(\theta)\cos(\theta)}{m\sin^2(\theta)+M}\right)$$

$$\ddot{X} = -g\left(\frac{m\sin(\theta)\cos(\theta)}{m\sin^2(\theta)+M}\right)$$

We can now find the total acceleration of each component. Since there is no acceleration in the vertical direction of the ramp, it's pretty easy,

$$A = \ddot{X} = -g\left(\frac{m\sin(\theta)\cos(\theta)}{m\sin^2(\theta)+M}\right)$$

The block,

$$a^2 = \ddot{x}^2 + \ddot{y}^2 = \left(\frac{g^2}{(m\sin^2 + M)^2}\right)[M^2\sin^2(\theta)\cos^2(\theta) + (m+M)^2\sin^4(\theta)]$$

$$= \left(\frac{g^2}{(m\sin^2 + M)^2}\right)\sin^2(\theta)(M + m\sin^2(\theta))^2$$

$$= g^2\sin^2(\theta)$$

$$a = g\sin(\theta)$$

1.3 Variations on a Rolling Cylinder

Figures 1.12(a)-(e) show a hand pulling a circular cylindrical object (whose mass is distributed with cylindrical symmetry). The cylinder has radius R, mass M, and moment of inertia I about its symmetry axis. The hand applies a force F by means of a weightless, flexible string. In all four cases find the acceleration A of the center of mass and the angular acceleration α of the cylindrical object; show explicitly that the work-energy theorem is satisfied.

1.3.a Empty space, no gravity, the string passes through the center of mass of the cylinder.

In this case, we have no angular acceleration since there is no friction to cause the cylinder to roll. In fact, this problem would be the same regardless of the shape of the cylinder. The only force is from the string,

$$A = \frac{F}{M}$$

To verify that the work-energy theorem (1.8), we can find the work,

$$W = Fx$$

Using the equations of kinematics, we can substitute in the linear acceleration,

$$= (MA) \cdot \left(1/2\ At^2\right) = 1/2\ M(At)^2$$

$$W = 1/2\ Mv^2$$

We recognize the right side as the kinetic energy of an object.

1.3.b Empty space, no gravity, the string is wrapped around the cylinder. [Question: How can the hand, applying the same force as in Part (a), supply the (hint) same translational kinetic energy as in Part (a) plus the extra rotational kinetic energy?]

This time, we have the same linear acceleration as before, but now we add an additional rotational component.

$$A = \frac{F}{M}$$

$$\alpha = \frac{FR}{I}$$

Same as before, to find the work we multiply the force by the total distance traveled,

$$W = F(x + R\theta)$$

$$= (MA)\left(1/2\ At^2\right) + R\left(\frac{I\alpha}{R}\right)\left(1/2\ \alpha t^2\right)$$

$$= 1/2\ M(At)^2 + 1/2\ I(\alpha t)^2$$

$$W = 1/2\ Mv^2 + 1/2 I\omega^2$$

The work is made up on the linear kinetic energy and the rotational kinetic energy.

To answer the additional question, think of a door. If you push the door near the hinges, it is a lot more difficult to close than if you apply the force near the handle. Then imagine that you take the door off the hinges. Now it doesn't matter where you push the door, it is going to move the same linear distance.

1.3.c Uniform vertical gravitation sufficient, together with friction, to constrain the cylinder to roll without slipping on the surface shown. The string passes through the center of mass of the cylinder.

Now we introduce friction, which opposes the motion of the cylinder. In the linear direction,

$$F - f = MA$$

In the angular direction,

$$fR = I\alpha$$

The condition for rolling without slipping is $\alpha R = A$.We can solve for A and α,

$$F - \frac{I\alpha}{R} = MA$$

$$F - \frac{IA}{R^2} = MA$$

$$FR^2 = A(MR^2 + I)$$

$$A = \frac{FR^2}{MR^2 + I}$$

$$\alpha = \frac{FR}{MR^2 + I}$$

As we did in the previous part, the work is a combination of the linear distance moved and the angular distance,

$$W = (F - f)x + fR\theta$$

$$= (F - f)\left(\frac{1}{2}\frac{F - f}{M}t^2\right) + R\left(\frac{I\alpha}{R}\right)\left({}^1\!/_2\ I\alpha t^2\right)$$

$$= {}^1\!/_2\ M\left(\frac{F - f}{M}t\right)^2 + {}^1\!/_2\ I(\alpha t)^2$$

$$W = {}^1\!/_2\ Mv^2 + {}^1\!/_2\ I\omega^2$$

1.3.d Same as Part (c), but with the string wrapped around the cylinder. [Questions: In which direction is the frictional force? How does the hand manage to supply the necessary translational and kinetic energies different from Part (c)?]

Let's go ahead and answer the additional questions first. If we had no friction, we would expect the wheel to rotate clockwise, so the frictional force must point counterclockwise since it has to oppose that motion. I think the answer to the second additional question is similar to that for part (b).

As in part (c), let's write down the equations of motion,

$$F - f = MA$$

$$(F + f)R = I\alpha$$

Using the rolling without slipping condition, we can solve for A and α.

$$A = \frac{2FR^2}{MR^2 + I}$$

$$\alpha = \frac{2FR}{MR^2 + I}$$

The work,

$$W = (F - f)x + (F - f)R\theta$$

$$= (F - f)\left(\frac{1}{2}\frac{F-f}{M}t^2\right) + (I\alpha)\left(1/2\ I\alpha t^2\right) \tag{1.3.1}$$

$$= 1/2\ M\left(\frac{F-f}{M}t\right)^2 + 1/2\ I(\alpha t)^2$$

$$W = 1/2\ Mv^2 + 1/2\ I\omega^2$$

1.3.e Same as Part (c), but with the string now wrapped around a shaft of radius $r < R$ within the cylinder (it's a kind of yo-yo). [Question: In which direction is the frictional force?]

This time, without friction, the wheel would rotate counter-clockwise, so the frictional force must point clockwise. Our equations of motion are similar to before,

$$F - f = MA$$

$$fR - Fr = I\alpha$$

Using the rolling without slipping condition, we can solve,

$$A = \frac{F(R^2 - rR)}{MR^2 + I}$$

$$\alpha = \frac{F(R - r)}{MR^2 + I}$$

The work,

$$W = (F - f)x + fR\theta - Fr\theta$$

$$= (MA)\left(1/2\ At^2\right) + (I\alpha)\left(1/2\ \alpha t^2\right)$$

$$= 1/2\ M(At)^2 + 1/2\ I(\alpha t)^2$$

$$W = 1/2\ Mv^2 + 1/2\ I\omega^2$$

1.4 Elastic Collision

A particle of mass m_1 makes an elastic (kinetic-energy conserving) collision with another particle of mass m_2. Before the collision m_1 has velocity $\vec{v}_1$ and m_2 is at rest relative to a certain inertial frame which we shall call the laboratory system. After the collision m_1 has velocity $\vec{u}_1$ making an angle θ with $\vec{v}_1$.

1.4.a Find the magnitude of $\vec{u}_1$

From conservation of momentum,

$$m_1\vec{v}_1 = m_1\vec{u}_1 + m_2\vec{u}_2$$

From conservation of energy,

$$m_1 v_1^2 = m_1 u_1^2 + m_2 u_2^2$$

We rearrange the conservation of momentum equation and square both sides. We want to rearrange since we know the angle between $\vec{v}_1$ and $\vec{u}_1$, but we don't know the angle between $\vec{u}_1$ and $\vec{u}_2$,

$$m_1\vec{v}_1 - m_1\vec{u}_1 = m_2\vec{u}_2$$

$$m_1^2 v_1^2 + m_1^2 u_1^2 - 2m_1^2 u_1 v_1 \cos(\theta) = m_2^2 u_2^2$$

Substituting u_2 into the conservation of energy equation,

$$m_1 v_1^2 - m_1 u_1^2 - \frac{m_1^2}{m_2}(v_1^2 + u_1^2 - 2u_1 v_1 \cos(\theta)) = 0$$

$$u_1^2(m_1 m_2 + m_1^2) - 2m_1^2 u_1 v_1 \cos(\theta) + v_1^2(m_1^2 - m_1 m_2) = 0$$

Using the quadratic equation,

$$u_1 = v_1 \frac{m_1 \cos(\theta) \pm \sqrt{m_2^2 - m_1^2 \sin^2(\theta)}}{m_1 + m_2}$$

To determine if we want plus or minus, we look at the limiting case $m_2 = m_1$. We expect that when two balls of equal mass collide, m_2 should continue moving in a straight line while m_1 stops moving. This is why you want to hit pool balls on the bottom since that gives the ball no spin and causes it to stop moving when it hits another ball (I think, I am rather rubbish at pool).

$$u_1 = v_1 \frac{m \cos(\theta) \pm \sqrt{m^2 - m^2 \sin^2(\theta)}}{2m}$$

$$= v_1 \frac{m\cos(\theta) \pm \sqrt{m^2\cos^2(\theta)}}{2m}$$

Since we want u_1 to vanish, we choose the negative,

$$u_1 = v_1 \frac{m_1\cos(\theta) - \sqrt{m_2^2 - m_1^2\sin^2(\theta)}}{m_1 + m_2}$$

1.4.b Relative to another inertial frame, called the center-of-mass system, the total linear momentum of the two-body system is zero. Find the velocity of the center-of-mass system relative to the laboratory system.

Let's say that the center-of-mass (CoM) is moving with velocity V. To convert from the lab frame to the CoM frame,

$$\begin{cases} v_1' = v_1 - V \\ v_2' = -V \end{cases}$$

Since the momentum in the CoM frame is zero,

$$m_1 v_1' + m_2 v_2' = 0$$

$$m_1(v_1 - V) - m_2 V = 0$$

$$m_1 v_1 = (m_1 + m_2)V$$

$$V = \frac{m_1}{m_1 + m_2} v_1$$

1.4.c Find the velocities $\vec{v}_1'$, $\vec{v}_2'$, $\vec{u}_1'$, $\vec{u}_2'$ of the two bodies before and after the collision in the center-of-mass system. Find the scattering angle θ' (the angle between $\vec{v}_1'$ and $\vec{u}_1'$) in terms of θ.

Solving for v_1' and v_2',

$$\begin{cases} v_1' = v_1 - V \\ v_2' = -V \end{cases}$$

$$\begin{cases} \vec{v}_1' = \dfrac{m_2}{m_1+m_2} v_1 \\ \vec{v}_2' = -\dfrac{m_1}{m_1+m_2} v_1 \end{cases}$$

From conservation of momentum in the CoM frame,

$$m_1\vec{v}_1' + m_2\vec{v}_2' = m_1\vec{u}_1' + m_2\vec{u}_2' = 0$$

If we want to convert from the lab frame to the CoM frame,

$$\vec{u}_1' = \vec{u}_1 - \vec{V}$$

We want to set $\vec{v}_1$ to lie along the x-axis. What this means is we can write,

$$\vec{u}_1 = u_1(\cos(\theta), \sin(\theta))$$

$$\vec{u}_1' = (u_1\cos(\theta) - V, u_1\sin(\theta))$$

From the conservation of momentum equation,

$$\vec{u}_2' = -\frac{m_1}{m_2}\vec{u}_1'$$

To find the scattering angle,

$$\cot(\theta') = \frac{u_{1x}'}{u_{1y}'} = \frac{u_{1x} - V}{u_{1y}}$$

$$= \frac{u_{1x}}{u_{1y}} - \frac{V}{u_{1y}}$$

The first term we recognize as being related to the scattering angle in the lab frame,

$$= \cot(\theta) - \frac{m_1 v_1}{(m_1+m_2)u_1\sin(\theta)}$$

$$\cot(\theta') = \cot(\theta) - \frac{m_1}{\sin(\theta)\left(m_1\cos(\theta) - \sqrt{m_2^2 - m_1^2\sin^2(\theta)}\right)}$$

1.5 Two-Body Problem

Two masses m_1 and m_2 in a uniform gravitational field are connected by a spring of unstretched length h and spring constant k. The system is held by m_1 so that m_2 hangs down vertically, stretching the spring. At $t = 0$ both m_1 and m_2 are at rest, and m_1 is released, so that the system starts to fall. Set up a suitable coordinate system and describe the subsequent motion of m_1 and m_2.

We have two equations of motion: one for the center of mass and one for the spring. For the center of mass, the only force on it is the gravitational force,

$$\ddot{X} = -g$$

We can solve for the center of mass using equation (1.14),

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

For the spring, we need to use the reduced mass (1.22) and the distance between the two points, x,

$$\mu \ddot{x} = -kx$$

Integrating the center of mass equation,

$$X = -{}^1\!/\!_2 g t^2 + v_0 t + X_0$$

For the spring, we use the solution to a simple harmonic oscillator,

$$x = A\cos(\omega t)$$

$$\omega^2 = \frac{k}{\mu}$$

Our initial conditions,

$$\begin{cases} x_1(0) = 0 \\ x_2(0) = h + \dfrac{m_2 g}{k} \\ \dot{x}_1(0) = \dot{x}_2(0) = 0 \end{cases}$$

Substituting these in,

$$x_2(0) - x_1(0) = A\cos(0)$$

$$A = h + \frac{m_2 g}{k}$$

$$v_0 = 0$$

$$X(0) = X_0$$

$$\frac{m_2}{m_1 + m_2}\left(h + \frac{m_2 g}{k}\right) = X_0$$

Our equations of motion,

$$\begin{cases} x(t) = A\cos(\omega t) \\ X(t) == -{}^1\!/\!{}_2 gt^2 + \dfrac{m_2 A}{m_1 + m_2} \end{cases}$$

Let's write these in terms of x_2 and x_1,

$$\begin{cases} x_2 - x_1 = A\cos(\omega t) \\ \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = -{}^1\!/\!{}_2 gt^2 + \dfrac{m_2 A}{m_1 + m_2} \end{cases}$$

Solving for x_1,

$$m_1 x_1 + m_2 x_2 = -{}^1\!/\!{}_2 (m_1 + m_2) gt^2 + m_2 A$$

$$m_1 x_1 + m_2 (A\cos(\omega t) + x_1) = -{}^1\!/\!{}_2 \ (m_1 + m_2) gt^2 + m_2 A$$

$$x_1 (m_1 + m_2) = -{}^1\!/\!{}_2 (m_1 + m_2) gt^2 + m_2 A (1 - \cos(\omega t))$$

$$x_1 = -{}^1\!/\!{}_2 \ gt^2 + \frac{m_2 A}{m_1 + m_2}(1 - \cos(\omega t))$$

For x_2,

$$x_2 = x_1 + A\cos(\omega t)$$

$$x_2 = -{}^1\!/\!{}_2 \ gt^2 + \frac{A}{m_1 + m_2}(m_2 + m_1 \cos(\omega t))$$

1.6 Three-Body Problem

The Earth and the Moon form a two-body system interacting through their mutual gravitational attraction. IN addition, each body is attracted by the gravitational field of the Sun, which in the sense of Section 1.3 is an external force. Take the Sun as the origin and write down the equations of motion for the center of mass $\vec{X}$ and the relative position $\vec{x}$ of the Earth-Moon system. Expand the resulting expressions in powers of x/X, the ratio of the magnitudes. Show that to lowest order in x/X the center of mass and relative position are uncoupled, but that in higher orders they are coupled because the Sun's gravitational force is not constant.

From the center of mass equation (1.14) and the relative distances, the equations of motion are,

$$\begin{cases} \ddot{\vec{X}} = \dfrac{m_e\ddot{\vec{r}}_e + m_m\ddot{\vec{r}}_m}{M} \\ \\ \ddot{\vec{x}} = \ddot{\vec{r}}_m - \ddot{\vec{r}}_e \end{cases}$$

Using the gravitational force equation,

$$\begin{cases} \ddot{\vec{r}}_e = -\dfrac{Gm_s}{r_e^3}\vec{r}_e + \dfrac{Gm_m}{x^3}\vec{x} \\ \\ \ddot{\vec{r}}_m = -\dfrac{Gm_s}{r_m^3}\vec{r}_m - \dfrac{Gm_e}{x^3}\vec{x} \end{cases}$$

Substituting these into the equations of motion,

$$\ddot{\vec{X}} = -\frac{1}{M}\left(-\frac{Gm_sm_e}{r_e^3}\vec{r}_e + \frac{Gm_mm_e}{x^3}\vec{x} - \frac{Gm_sm_m}{r_m^3}\vec{r}_m - \frac{Gm_em_m}{x^3}\vec{x}\right)$$

$$\ddot{\vec{X}} = -\frac{Gm_s}{M}\left(\frac{m_e}{r_e^3}\vec{r}_e + \frac{m_m}{r_m^3}\vec{r}_m\right)$$

$$\ddot{\vec{x}} = -\frac{Gm_s}{r_m^3}\vec{r}_m - \frac{Gm_e}{x^3}\vec{x} + \frac{Gm_s}{r_e^3}\vec{r}_e - \frac{Gm_m}{x^3}\vec{x}$$

$$\ddot{\vec{x}} = -Gm_s\left(\frac{\vec{r}_m}{r_m^3} - \frac{\vec{r}_e}{r_e^3}\right) - \frac{GM}{x^3}\vec{x}$$

We want to write $\vec{r}_e$ in terms of $\vec{x}$ and $\vec{X}$ so we can write our equations of motion solely using $\vec{x}$ and $\vec{X}$,

$$\vec{r}_m = \vec{x} + \vec{r}_e$$

$$M\vec{X} = m_e\vec{r}_e + m_m\vec{r}_m$$

$$= m_e\vec{r}_e + m_m(\vec{x} + \vec{r}_e)$$

$$= M\vec{r}_e + m_m\vec{x}$$

$$\vec{r}_e = \vec{X} - \frac{m_m}{M}\vec{x}$$

We can do the same for $\vec{r}_m$,

$$\vec{r}_m = \vec{X} + \frac{m_e}{M}\vec{x}$$

We notice we have r_e^{-3} and r_m^{-3}. Rather than writing these out explicitly, we can use approximations,

$$r_e^{-3} = \left[\left(\vec{X} - \frac{m_m}{M}\vec{x}\right)\left(\vec{X} - \frac{m_m}{M}\vec{x}\right)\right]^{-3/2}$$

$$= \left[X^2 - 2\frac{m_m}{M}\vec{X}\cdot\vec{x} + \left(\frac{m_m}{M}\right)^2 x^2\right]^{-3/2}$$

$$= \frac{1}{X^3}\left[1 + \left(-2\frac{m_m}{M}\frac{\vec{X}\cdot\vec{x}}{X^2} + \left(\frac{m_m}{M}\right)^2\frac{x^2}{X^2}\right)\right]^{-3/2}$$

Using the binomial approximation,

$$r_e^{-3} \approx \frac{1}{X^3}\left[1 - \frac{3}{2}\left(-\frac{2m_m}{M}\frac{\vec{X}\cdot\vec{x}}{X^2} + \left(\frac{m_m}{M}\right)^2\frac{x^2}{X^2}\right)\right]$$

Similarly,

$$r_m^{-3} \approx \frac{1}{X^3}\left[1 - \frac{3}{2}\left(-\frac{2m_e}{M}\frac{\vec{X}\cdot\vec{x}}{X^2} + \left(\frac{m_e}{M}\right)^2\frac{x^2}{X^2}\right)\right]$$

If we're keeping the lowest order in x/X, we only keep the first terms,

$$r_m^{-3} \approx r_e^{-3} \approx \frac{1}{X^3}$$

Substituting into $\ddot{\vec{x}}$,

$$\ddot{\vec{x}} = -\frac{Gm_s}{X^3}\vec{x} - \frac{GM}{x^3}\vec{x}$$

$$\ddot{\vec{x}} \approx -\frac{GM}{x^3}\vec{x}$$

For $\ddot{\vec{X}}$,

$$\ddot{\vec{X}} = -\frac{Gm_s}{MX^3}(M\vec{X})$$

$$\ddot{\vec{X}} \approx -\frac{Gm_s}{X^3}\vec{X}$$

1.7 Particle in Polynomial Potential

Show that a one-dimensional particle subject to the force $F = -kx^{2n+1}$, where n is an integer, will oscillate with a period proportional to A^{-n}, where A is the amplitude. Pay special attention to the case of $n \leq 0$.

Let's go ahead and work with the potential(1.10),

$$V = \frac{k}{2n+2}x^{2n+2}$$

From quadrature (1.12), the period is given by,

$$P = \sqrt{2m}\int_{-A}^{A}\frac{dx}{\sqrt{E-V}}$$

We know that the energy is equal to the maximum value of the potential, i.e., $E = V(A)$.

$$= \sqrt{\frac{2m(2n+2)}{k}}\int_{-A}^{A}\frac{dx}{A^{2n+2}-x^{2n+2}}$$

Setting $u = x/A$,

$$= \sqrt{\frac{2m(2n+2)}{k}}\int_{-1}^{1}\frac{A\ du}{\sqrt{A^{2n+2}+(Au)^{2n+2}}}$$

$$= \sqrt{\frac{2m(2n+2)}{k}}\frac{1}{A^n}\int_{-1}^{1}\frac{du}{1-u^{2n+2}}$$

The integral will be left to the reader as an exercise, but we don't actually need to solve it since it has no dependence on A. From this, we can see that the period goes by A^{-n} as desired.

Now what happens if we have a negative n. Specifically, let's look at the case where $n = -1$. In this case, $F = -kx^{-1}$, so the potential becomes $V = k\ln(x)$. The period,

$$P = \sqrt{\frac{2m}{k}}\int_{-A}^{A}\frac{dx}{\sqrt{\ln(A)-\ln(x)}}$$

$$= \sqrt{\frac{2m}{k}}\int_{-A}^{A}\frac{dx}{\sqrt{-\ln(x/A)}}$$

Setting $u = x/A$,

$$P = \sqrt{\frac{2m}{k}}A\int_{-1}^{1}\frac{du}{\sqrt{-\ln(u)}}$$

Once again the integral does not depend on A, so the period goes by A as expected.

1.8 Yo-yo Motion

A yo-yo consists of two disks of mass M and radius R connected by a shaft of mass m and radius r; a weightless string is wrapped around the shaft.

1.8.a The free end of the string is held stationary in the Earth's gravitational field. Assuming that the string starts out vertical, find the motion of the yo-yo's center of mass.

We start by writing the Lagrangian,

$$\mathscr{L} = \frac{1}{2}\mu\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{r^2} + \mu g x$$

$$\mu = 2M + m$$

$$I = MR^2 + \frac{1}{2}mr^2$$

Note that the outer disks will have the same rotational velocity since they are connected. Using the Lagrangian equations of motion,

$$\mu g = \frac{d}{dt}\left[\mu\dot{x} + \frac{I}{r^2}\dot{x}\right]$$

$$\ddot{x} = \frac{\mu g}{\mu + I/r^2}$$

1.8.b The free end is moved so as to keep the yo-yo's center of mass stationary. Describe the motion of the free end of the string and the rotation of the yo-yo.

Going to force diagrams, the tension must be equal to the gravitational force,

$$T = \mu g$$

We can then use torque to find the rotation of the yo-yo.

$$Fr = I\alpha$$

$$\alpha = \frac{\mu g r}{I}$$

The free end must also obey this acceleration,

$$a = \alpha r = \frac{\mu g r^2}{I}$$

1.8.c The yo-yo is transported to empty space, where there is no gravitational field, and a force F is applied to the free end of the string. Describe the motion of the center of mass of the yo-yo, the yo-yo's rotation, and the motion of the free end of the string.

As again, if there is an applied force on the free end, the center of mass must feel the same force,

$$a = \frac{F}{\mu}$$

$$\alpha = \frac{Fr}{I}$$

The free end will accelerate as a combination of both the center of mass and the rotation,

$$A = a + \alpha r = F\left(\frac{1}{\mu} + \frac{r^2}{I}\right)$$

1.9 Terminal Velocity

A particle in a uniform gravitational field experiences an additional retarding force $F = -\alpha\vec{v}$, where $\vec{v}$ is its velocity. Find the general solution to the equations of motion and show that the velocity has an asymptotic value (called the terminal velocity). Find the terminal velocity.

We start by writing the force in this field,

$$\vec{F} = m\vec{g} - \alpha\vec{v}$$

We can effectively treat this as a one-dimensional problem, so let's get rid of those vector,

$$m\ddot{x} = mg - \alpha\dot{x}$$

I don't like solving second-order differential equations, so let's turn it into a first-order differential equation,

$$m\dot{v} = mg - \alpha v$$

$$m\frac{dv}{dt} = mg - \alpha v$$

I had to look up how to solve non-homogeneous first-order differential equations,

$$\left[\frac{dv}{dt} + \frac{\alpha}{m}v = g\right]\exp\left(\frac{\alpha t}{m}\right)$$

$$\exp\left(\frac{\alpha t}{m}\right)\frac{dv}{dt} + \frac{\alpha}{m}\exp\left(\frac{\alpha t}{m}\right)v = g\exp\left(\frac{\alpha t}{m}\right)$$

We use the product rule on the left side,

$$\frac{d}{dt}\left[\exp\left(\frac{\alpha t}{m}\right)v\right] = g\exp\left(\frac{\alpha t}{m}\right)$$

$$\exp\left(\frac{\alpha t}{m}\right)v = g\exp\left(\frac{\alpha t}{m}\right)\ dt$$

$$v = \frac{mg}{\alpha} + c\exp\left(-\frac{\alpha t}{m}\right)$$

Setting the initial velocity to v_0, we can solve for the constant, leaving us with the full solution,

$$v(t) = \frac{mg}{\alpha} - \left(\frac{mg}{\alpha} - v_0\right)\exp\left(-\frac{\alpha t}{m}\right)$$

As time goes to infinity, the exponential term dies, which means the velocity has some asymptotic behaviour. It approaches $v = mg/\alpha$. To find the position, we can integrate over time and set $x(0) = x_0$,

$$x(t) = \frac{mg}{\alpha}t + \left(\frac{m}{\alpha}\right)\left(\frac{mg}{\alpha} - v_0\right)\left[\exp\left(-\frac{\alpha t}{m}\right)\right] + c$$

$$x(t) = \frac{mg}{\alpha}t + \left(\frac{m}{\alpha}\right)\left(\frac{mg}{\alpha} - v_0\right)\left[\exp\left(-\frac{\alpha t}{m}\right) - 1\right] + x_0$$

1.10 Trajectory Derivation

Change the variable of integration in Eq. (1.7) from s to any other parameter in order to show that the distance between two points on the trajectory, as defined by (1.7), is indeed independent of the parameter.

Jose's equation (1.7),

$$l(s_0, s_1) = \int_{s_0}^{s_1} \left(\frac{dx_i}{ds} \frac{dx_i}{dx} \right)^{1/2} ds$$

We'll perform a change of variables by changing $s \to \alpha$. Further, we say that $\alpha(s_0) = \alpha_0$ and $\alpha(s_1) = \alpha_1$.

$$l \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{d\alpha}{ds} \cdot \frac{dx_i}{d\alpha} \frac{d\alpha}{ds} \right)^{1/2} \frac{ds}{d\alpha} \, d\alpha$$

$$= \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{dx_i}{d\alpha} \right)^{1/2} \frac{d\alpha}{ds} \frac{ds}{d\alpha} \, d\alpha$$

$$= \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{dx_i}{d\alpha} \right)^{1/2} d\alpha$$

Which is the same form as the equation in the problem, so the integral is independent of the parameter.

1.11 Curvature and the Frenet Formulas

1.11.a The concept of curvature and radius of curvature are defined by extending those concepts from circles to curves in general. The curvature κ is defined, as in Eq. (1.13), as the rate (with respect to length along the curve) of rotation of the tangent vector. Show that what Eq. (1.13) defines is in fact the rate of rotation of τ (i.e., that it gives the rate of change of the angle τ makes with a fixed direction). Show also that for a circle in the plane $\kappa = 1/R$, where R is the radius of the circle.

Jose's equation (1.13),

$$\lim_{t_1 \to t_2} \frac{|\vec{\tau}(t_1) - \vec{\tau}(t_2)|}{|l(t_1) - l(t_2)|} = \left| \frac{d\vec{\tau}}{dl} \right| = \kappa$$

We start by taking the scalar product of $\vec{\tau}$ and a $\hat{N}$, a fixed unit vector. If we take the derivative according l,

$$\frac{d}{dl}(\vec{\tau} \cdot \hat{N}) = \frac{d\vec{\tau}}{dl} \cdot \hat{N}$$

$$= \left| \frac{d\vec{\tau}}{dl} \right| \cos(\phi)$$

Alternatively, we could apply the scalar product before taking the derivative,

$$\frac{d}{dl}(\vec{\tau} \cdot \hat{N}) = \frac{d}{dl}\left(\tau \cos(\theta)\right)$$

$$= -\tau \sin(\theta) \frac{d\theta}{dl}$$

From the text, we know that $\frac{d\vec{\tau}}{dt}$ and $\vec{\tau}$ are orthogonal, so θ and ϕ are off by a factor of $\pi/2$, which means $\sin(\theta) = -\cos(\phi)$. Setting the two solutions equal to each other,

$$\left| \frac{d\vec{\tau}}{dl} \right| \cos(\phi) = -\tau \sin(\theta) \frac{d\theta}{dl}$$

$$\left| \frac{d\vec{\tau}}{dl} \right| = \tau \frac{d\theta}{dl}$$

If we then normalize $\vec{\tau}$,

$$\left| \frac{d\vec{\tau}}{dl} \right| = \kappa = \frac{d\theta}{dl}$$

For a circle, $l = R\theta$, so $\kappa = 1/R$.

1.11.b Derive the second of the Frenet formulas from the fact that $\hat{\tau}$, $\hat{n}$, and $\hat{B}$ are a set of orthogonal unit vectors and from the definition of θ.

The second Frenet formula is

$$\dot{\hat{n}} = -\kappa \dot{l} \vec{\tau} + \theta \dot{l} \hat{B}$$

We'll start by looking at the time derivative of $\hat{B}$,

$$\dot{\hat{B}} = \dot{\hat{\tau}} \times \hat{n} + \hat{\tau} \times \dot{\hat{n}}$$

The first term dies because of the first Frenet formula ($\hat{n} \times \hat{n} = 0$). In addition, from the text, $\dot{\hat{B}} = -\theta \dot{l} \hat{n}$,

$$-\theta \dot{l} \hat{n} = \hat{\tau} \times \dot{\hat{n}}$$

If we look at the term,

$$\theta \dot{l} \hat{B} = \theta \dot{l} (\hat{\tau} \times \hat{n})$$

Using the definition of $\dot{\hat{B}}$ from the text,

$$= \tau \times (\dot{\hat{n}} \times \hat{\tau})$$

Using the vector triple product (BACCAB),

$$= \dot{\hat{n}} - \tau (\dot{\hat{n}} \cdot \tau)$$

Since $\hat{n}$ and $\hat{\tau}$ are orthogonal, the time derivative of their scalar product must also be equal to 0, so $\dot{\hat{n}} \cdot \hat{\tau} = -\hat{n} \cdot \dot{\hat{\tau}}$. Further, using the first Frenet formula, $\dot{\hat{n}} \cdot \hat{\tau} = -\kappa \dot{l}$. Substituting this in,

$$\theta \dot{l} \hat{B} = \dot{\hat{n}} + \kappa \dot{l} \hat{\tau}$$

$$\dot{\hat{n}} = -\kappa \dot{l} \hat{\tau} + \theta \dot{l} \hat{B}$$

1.12 Particle on an Ellipse

A particle is constrained to move at constant speed on the ellipse $a_{ij}x^i x^j = 1 (i, j = 1, 2)$. Find the Cartesian components of its acceleration as a function of position on the ellipse.

Since it is moving in a curve, the acceleration of the particle is,

$$a = \frac{v^2}{R}$$

where, by looking this up,

$$R = \frac{(1 + y'^2)^{3/2}}{y''}$$

From the equation for an ellipse with semimajor axis A along the x-axis and semiminor axis B along the y-axis,

$$y = \frac{B}{A}(A^2 - x^2)^{1/2}$$

$$y' = -\frac{B}{A}x(A^2 - x^2)^{-1/2}$$

$$y'' = -\frac{BA}{(A^2 - x^2)^{3/2}}$$

Substituting these in,

$$R = \frac{\left(1 + \frac{B^2x^2}{A^2(A^2 - x^2)}\right)^{3/2}}{-\frac{BA}{(A^2 - x^2)^{3/2}}}$$

$$= -\frac{[A^2(A^2 - x^2) + B^2x^2]^{3/2}}{BA^4}$$

$$R = -\frac{[A^4 + (B^2 - A^2)x^2]^{3/2}}{A^4B}$$

From this, the total acceleration,

$$a = -v^2\frac{A^4B}{[A^4 + (B^2 - A^2)x^2]^{3/2}}$$

The components can be found by looking them up. For the x-component,

$$a_x = a\sin(\theta) = ay'(1+y'^2)^{-1/2}$$

$$= -v^2 \frac{A^4 B}{[A^4 + (B^2 - A^2)x^2]^{3/2}} \left(-\frac{B}{A}x(A^2-x^2)^{-1/2}\right) \left(\frac{A^2(A^2-x^2)}{A^4+(B^2-A^2)x^2}\right)^{1/2}$$

$$a_x = \frac{v^2 A^4 B^2 x}{[A^4 + (B^2 - A^2)x^2]^2}$$

The y-component,

$$a_y = a\cos(\theta) = a(1+y'^2)^{-1/2}$$

We could repeat the process above, but we could also notice,

$$a_y = \frac{a_x}{y'} = \frac{v^2 A^4 B^2 x}{[A^4 + (B^2 - A^2)x^2]^2} \left(-\frac{A\sqrt{A^2-x^2}}{Bx}\right)$$

$$a_y = -\frac{v^2 A^5 B\sqrt{A^2-x^2}}{[A^4 + (B^2 - A^2)x^2]^2}$$

1.13 Existence of Mass

Show that if Eq. (1.17) is satisfied, there exists constants m_1, m_2, and m_3 such that Eqs. (1.15) and (1.16) can be put in the form of (1.18).

Eq. (1.15),

$$\vec{v}_1(t) + \mu_{12}\vec{v}_2(t) = \vec{K}$$

Eq. (1.16),

$$\begin{cases} \vec{v}_2(t) + \mu_{23}\vec{v}_3(t) = \vec{L} \\ \vec{v}_3(t) + \mu_{31}\vec{v}_1(t) = \vec{M} \end{cases}$$

Eq. (1.17),

$$\mu_{12}\mu_{23}\mu_{31} = 1$$

Eq. (1.18),

$$\begin{cases} m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{P}_{12} \\ m_2\vec{v}_2 + m_3\vec{v}_3 = \vec{P}_{23} \\ m_3\vec{v}_3 + m_1\vec{v}_1 = \vec{P}_{31} \end{cases}$$

We'll start with

$$\vec{v}_1 + \mu_{12}\vec{v}_2 = \vec{K}$$

We want to choose a μ_{12} such that we get

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{P}_{12}$$

One such choice is,

$$\begin{cases} \mu_{12} = \dfrac{m_2}{m_1} \\ \\ \vec{P}_{12} = m_1\vec{K} \end{cases}$$

We can do similar for the other μ,

$$\begin{cases} \mu_{23} = \dfrac{m_3}{m_2} \\ \\ \mu_{31} = \dfrac{m_1}{m_3} \end{cases}$$

1.14 Non-Inertial Frames

Consider Eq. (1.23) in two rather than three dimensions, and assume that the $\vec{x}$ and $\vec{y}$ coordinates are not both inertial, but rotating with respect to each other: $y_1 = x_1\cos(\omega t) - x_2\sin(\omega t)$, $y_2 = x_1\sin(\omega t) + x_2\cos(\omega t)$. Show that in general even if the $\vec{x}$ acceleration vanishes, the $\vec{y}$ acceleration does not. Find $\ddot{\vec{y}}$ for $\ddot{\vec{x}} = 0$, but $\dot{\vec{x}} \neq 0$ and $\vec{x} \neq 0$. Give the physical significance of the terms you obtain.

Eq. (1.23),

$$\begin{cases} y_i = f_i(x,t) \\ x_i = g_i(y,t) \end{cases}$$

We can write the given conditions in matrix notation,

$$|y\rangle = A\,|x\rangle$$

$$A = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Following the prescription in the text,

$$|\ddot{y}\rangle = \ddot{A}\,|x\rangle + 2\dot{A}\,|\dot{x}\rangle + A\,|\ddot{x}\rangle$$

If $|\ddot{x}\rangle = 0$, $|\ddot{y}\rangle$ does not necessarily vanish. We can find $|\ddot{y}\rangle$ by taking the necessary time derivatives. As we will see in a later chapter, the first term corresponds to centripetal acceleration and the second corresponds to Coriolis force,

$$\dot{A} = \begin{bmatrix} -\omega\sin(\omega t) & -\omega\cos(\omega t) \\ \omega\cos(\omega t) & -\omega\sin(\omega t) \end{bmatrix}$$

$$\ddot{A} = \begin{bmatrix} -\omega^2\cos(\omega t) & \omega^2\sin(\omega t) \\ -\omega^2\sin(\omega t) & -\omega^2\cos(\omega t) \end{bmatrix}$$

1.15 Particle in a Force Field

A particle of mass m moves in one dimension under the influence of the force

$$F = -kx + \frac{a}{x^3}$$

Find the equilibrium points, show that they are stable, and calculate the frequencies of oscillation about them. Show that the frequencies are independent of the energy.

The equilibrium points can be found by taking the first derivative of potential and setting that equal to 0. However, we're given force, which is already the negative derivative of the potential (1.10), so we're most of the way there. All we need to do is set the given force equal to 0 and solve for x,

$$-kx + \frac{a}{x^3} = 0$$

$$kx^4 = a$$

$$x = \pm\left(\frac{a}{k}\right)^{1/4}$$

To determine if these are stable, we take the second derivative of the potential (or the first derivative of the force and negatify it), plug in the equilibrium points and see if the result is positive,

$$-\frac{dF}{dx} = k + \frac{3a}{x^4}$$

$$= k + 3a\left(\frac{k}{a}\right) = 4k$$

This is positive as long as k is positive. Furthermore, this implies that a must also be positive since different signs would give imaginary solutions.

TO find the period of oscillation, we slightly perturb x, i.e., $x \to x + \epsilon$. Our original equation,

$$m\ddot{x} = -kx + \frac{a}{x^3}$$

becomes

$$m(\ddot{x} + \ddot{\epsilon}) = -k(x+\epsilon) + a(x+\epsilon)^{-3}$$

Using the binomial approximation on the second term,

$$= -kx - \epsilon k + ax^{-3}\left(1 - 3\frac{\epsilon}{x}\right)$$

$$m\ddot{x} + m\ddot{\epsilon} = -kx + \frac{a}{x^3} - \epsilon k - 3\frac{a\epsilon}{x^4}$$

$$m\ddot{\epsilon} = -\epsilon k - \frac{3a\epsilon}{x^4}$$

Substituting in the equilibrium point,

$$m\ddot{\epsilon} = -4k\epsilon$$

We recognize this as the simple harmonic oscillator, so the frequency is

$$\omega = 2\sqrt{\frac{k}{m}}$$

The frequency does not depend on the position or any other terms which might affect the energy.

1.16 Center of Mass, System of Particles

Consider a system of particles made of K subsystems, each itself a system of particles. Let M_I be the mass and $\vec{X}_I$ the center of mass of the Ith subsystem. Show that the center of mass of the entire system is given by an equation similar to (1.57), but with m_i and $\vec{x}_i$ replaced by M_I and $\vec{X}_I$ and the sum taken from $I = 1$ to $I = K$.

Jose (1.57),

$$\vec{X} = \frac{1}{M} \sum_i m_i \vec{x}_i$$

$$= \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i}$$

We can use this to define a single subsystem,

$$\vec{X}_I = \frac{1}{M_I} \sum_i m_i \vec{x}_i$$

If we then break up our subsystems into their constituent particles, we can define each particle with a mass m_{Ii} and $\vec{x}_{Ii}$ with the first index referring to the subsystem and the second index to the number of that particle within that subsystem. We then use (1.57) to write the total center of mass,

$$\vec{X} = \frac{\sum_I \sum_i m_{Ii} \vec{x}_{Ii}}{\sum_I \sum_i m_{Ii}}$$

We can regroup the particles back into their subsystems,

$$= \frac{1}{M} \sum_I M_I \vec{X}_I$$

Alternatively, we could think of each subsystem as a single particle and extrapolate from there.

1.17 Center of Mass, Kinetic Energy

Express the total kinetic energy of a system of N particles in terms of their center of mass and the relative positions of the particles [i.e, derive Eq. (1.65)]. Extend the result to a continuous distribution of particles with mass density $\rho(\vec{x})$. (Hint: Replace the sum by an integral)

Equation (1.65),

$$T = {}^1\!/\!_2\ M\dot{X}^2 + {}^1\!/\!_2\ \sum_i m_i \dot{y}_i^2$$

The kinetic energy of a system of particles (1.18) can be found by summing the kinetic energy of each individual particle. Before we do this, we should define the position of the ith particle relative to the center of mass,

$$\vec{y}_i = \vec{x}_i - \vec{X}$$

Now if we sum all the kinetic energies,

$$T = {}^1\!/\!_2\ \sum_i m_i \dot{x}_i^2$$

$$= {}^1\!/\!_2 \sum_i m_i (\dot{\vec{y}}_i + \dot{\vec{X}})^2$$

$$= {}^1\!/\!_2 \sum_i m_i \dot{y}_i^2 + \sum_i m_i \dot{\vec{y}}_i \cdot \dot{\vec{X}} + {}^1\!/\!_2 \sum_i m_i \dot{X}^2$$

The second term, we can kill by looking at the definition of center of mass (1.14),

$$\vec{X} = \frac{1}{M} \sum_i m_i \vec{x}_i = \frac{1}{M} \sum_i m_i (\vec{y}_i + \vec{X})$$

$$= \frac{1}{M} \sum_i m_i \vec{X} + \frac{1}{M} \sum_i m_i \vec{y}_i$$

$$\vec{X} = \vec{X} + \frac{1}{M} \sum_i m_i \vec{y}_i$$

If we sum over the relative positions, they should all cancel out. Similarly, if we sum over relative velocities, they should also cancel out. We are left with

$$T = {}^1\!/\!_2\ M\dot{X}^2 + {}^1\!/\!_2 \sum_i m_i \dot{y}_i^2$$

To convert to continuous distribution, we do as the hint suggests and convert the sum to an integral. We also convert the masses to mass density,

$$T = {}^1\!/\!_2\ M\dot{X}^2 + {}^1\!/\!_2 \int \rho \dot{y}^2\ d^3x$$

1.18 Internal Forces

In deriving Eq. (1.73) we assumed that the internal forces do not contribute to the total torque on a system of particles. Show explicitly that if for each i and j the internal force $\vec{F}_{ij}$ lies along the line connecting the ith and jth particles, then the internal forces indeed do not contribute to the total torque.

Equation (1.73),

$$\vec{N}_z = \dot{\vec{L}}_z$$

The total torque (1.6) can be found by summing over the individual torque of each particle,

$$\vec{N} = \sum_i \vec{z}_i \times \vec{F}_i$$

The internal torque is the torque between each particle,

$$\vec{N}_{int} = \sum_{i \neq j} \vec{z}_{ij} \times \vec{F}_{ij} = {}^1\!/_2 \sum_{i \neq j} (\vec{z}_i \times \vec{F}_{ij} + \vec{z}_j \times \vec{F}_{ji})$$

Using Newton's Third Law,

$$= {}^1\!/_2 \sum_{i \neq j} (\vec{z}_i \times \vec{F}_{ij} - \vec{z}_j \times \vec{F}_{ij})$$

$$= {}^1\!/_2 \sum_{i \neq j} (\vec{z}_i - \vec{z}_j) \times \vec{F}_{ij}$$

Since $\vec{z}_i - \vec{z}_j$ is in the same direction as $\vec{F}_{ij}$, the internal torque dies.

1.19 1.19

1.20 Stable Equilibrium

A particle of mass m moves along the x axis under the influence of the potential

$$V(x) = V_0 x^2 \exp(-ax^2)$$

where V_0 and $a > 0$ are constants. Find the equilibrium points of the motion, draw a rough graph of the potential, and draw the phase portrait of the system. On these graphs indicate the relation between the energy and geometry of the orbits in velocity phase space.

I'm just going to find the equilibrium points, which we do by taking the first derivative of the potential and setting it equal to 0.

$$\frac{dV}{dx} = 2V_0 x \exp(-ax^2) - 2V_0 a x^3 \exp(-ax^2) = 0$$

$$2V_0 x \exp(-ax^2) = 2V_0 a x^3 \exp(-ax^2)$$

$$\frac{1}{a} = x^2$$

$$x = \pm\sqrt{\frac{1}{a}}$$

1.21 1.21

1.22 1.22

1.23 1.23

1.24 1.24

1.25 1.25

1.26 Vector Product

Derive Eq. (1.85)

Equation (1.85),

$$\ddot{\vec{y}} = \omega \times (\omega \times \vec{y}) + 2\omega \times \dot{\vec{y}}$$

I believe there to be a typo here. I think this should be

$$\ddot{\vec{y}} = -\omega \times (\omega \times \vec{y}) + 2\omega \times \dot{\vec{y}}$$

Let's do this by brute force, defining

$$\begin{cases} \vec{y} = (y_1, y_2, y_3) \\ \vec{\omega} = (0, 0, \omega) \end{cases}$$

$$\vec{\omega} \times \vec{y} = (-\omega y_2, \omega y_1, 0)$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{y}) = (-\omega^2 y_1, -\omega^2 y_2, 0)$$

$$\omega \times \dot{\vec{y}} = (-\omega \dot{y}_2, \omega \dot{y}_1, 0)$$

Combining this, we can compare to equation (1.84),

$$\begin{cases} \ddot{y}_1 = \omega^2 y_1 - 2\omega \dot{y}_2 \\ \ddot{y}_2 = \omega^2 y_2 + 2\omega \dot{y}_1 \\ \ddot{y}_3 = 0 \end{cases}$$

1.27 Centrifugal Force

In the film "2001: A Space Odyssey" there is a toroidal space station rotating about a fixed axis that provides a centrifugal acceleration equal to the Earth's gravitational acceleration $g = 10m/s^2$ on a stationary object.

1.27.a Find the needed ω, assuming that the radius of the space station is 150m.

Using Jose (1.82) with $y_2 = y_3 = 0$,

$$y_1 = x_1 \cos(\omega t) - x_2 \sin(\omega t)$$

Jose (1.84) gives the acceleration,

$$\ddot{y}_1 = \omega^2 y_1$$

Note that because we set $y_2 = y_3 = 0$, we ignore some terms.

$$\omega^2 = \frac{\ddot{y}_1}{y_1} = \frac{10m/s^2}{150m}$$

$$\omega = 0.25s^{-1}$$

1.27.b Find the (fictitious) gravitational acceleration that would be felt by a person walking at $1.3m/s$ in two directions along the inner tube of the torus and across it.

A person walking along the inner tube will be moving parallel to the rotation, so there is no additional acceleration. A person walking across the inner tube has an additional term,

$$\dot{y}_2 = 1.3m/s$$

Using (1.84),

$$\ddot{y}_1 = \omega^2 y_1 - 2\omega\dot{y}_2$$

We've already established the first terms gives $10m/s^2$, so the second term,

$$-2\omega\dot{y}_2 = -2 * 0.25s^{-1} * 1.3m/s = -0.65m/s^2$$

Note this will either increase or decrease the centrifugal acceleration based on which direction they are moving.

1.27.c Find the (fictitious) acceleration that would be felt by a person sitting down or rising from a chair at 1.3m/s

This would be a velocity in y_1,

$$\dot{y}_1 = 1.3m/s$$

Looking at (1.84), this affects the acceleration in y_2,

$$\ddot{y}_2 = 2\omega\dot{y}_1 = 0.65m/s^2$$

Chapter 2

Lagrangian Formulation of Mechanics

2.1 Conservation of Energy

Show that if all the external forces are given by a time-independent potential (i.e., if $\vec{F}_i = -\nabla_i V(\vec{x}_1, ..., \vec{x}_N)$), then the rate of change of total energy is given by Eq. (2.15).

Jose equation (2.15),

$$\frac{dE}{dt} = -\sum_I \lambda_I \frac{\partial f_I}{\partial t}$$

As the book suggests, we should do the same thing as was done for the one-particle system. We start with

$$m\ddot{\vec{x}} \cdot \dot{\vec{x}} = \frac{d}{dt} \sum_i \frac{1}{2} m_i \dot{x}_i^2$$

$$= -\sum_i \nabla_i V_i \cdot \dot{\vec{x}}_i + \sum_{i,I} \nabla f_i \cdot \dot{\vec{x}}_i$$

We recognize this as the change in kinetic energy,

$$\frac{dT}{dt} = -\frac{dV}{dt} - \sum_I \lambda_i \frac{\partial f_I}{\partial t}$$

$$\frac{dE}{dt} = -\sum_I \lambda_I \frac{\partial f_I}{\partial t}$$

2.2 2.2

2.3 Coordinate Transformation

Prove that if $\mathscr{L}'$ is defined by (2.36), then Eq. (2.28) implies Eq. (2.37).

Jose equation(2.36),

$$\mathscr{L}'(q', \dot{q}', t) = \mathscr{L}(q(q', t), \dot{q}(q', \dot{q}', t), t) = \mathscr{L}(q, \dot{q}, t)$$

Jose equation(2.28),

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathscr{L}}{\partial q^\alpha} = 0$$

Jose equation(2.37),

$$\frac{d}{dt}\frac{\partial \mathscr{L}'}{\partial \dot{q}'^\alpha} - \frac{d\mathscr{L}'}{dq'^\alpha} = 0$$

We'll start with the last equation. Using the chain rule,

$$\frac{d}{dt}\frac{\partial \mathscr{L}'}{\partial \dot{q}'} - \frac{d\mathscr{L}'}{dq'} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \dot{q}'}\right) - \frac{\partial \mathscr{L}'}{\partial q}\frac{\partial q}{\partial q'} - \frac{\partial \mathscr{L}'}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q'}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}}\right)\frac{\partial \dot{q}}{\partial \dot{q}'} + \frac{\partial \mathscr{L}'}{\partial \dot{q}}\frac{d}{dt}\left(\frac{\partial \dot{q}}{\partial \dot{q}'}\right) - \frac{\partial \mathscr{L}'}{\partial q}\frac{\partial q}{\partial q'} - \frac{\partial \mathscr{L}'}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q'}$$

We can convince ourselves that

$$\begin{cases} \dfrac{\partial \dot{q}}{\partial q'} = \dfrac{d}{dt}\left(\dfrac{\partial q}{\partial q'}\right) \\[2ex] \dfrac{\partial \dot{q}}{\partial \dot{q}'} = \dfrac{\partial q}{\partial q'} \end{cases}$$

Using these,

$$\left[\frac{d}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}'}{\partial q}\right]\frac{\partial q}{\partial q'} + \frac{\partial \mathscr{L}'}{\partial \dot{q}}\left[\frac{d}{dt}\left(\frac{\partial \dot{q}}{\partial \dot{q}'}\right) - \frac{\partial \dot{q}}{\partial q'}\right]$$

The first term disappears using Jose equation (2.28), and the second disappears due to the corollaries we found.

2.4 2.4

2.5 2.5

2.6 2.6

2.7 2.7

2.8 2.8

2.9 Double Pendulum

A double plane pendulum consists of a simple pendulum (mass m_1, length l_1) with another simple pendulum (mass m_2, length l_2) suspended from m_1, both constrained to move in the same vertical plane.

2.9.a Describe the configuration manifold $\mathbb{Q}$ of this dynamical system. Say what you can about T$\mathbb{Q}$.

2.9.b Write down the Lagrangian of this system in suitable coordinates

If we define the deviation of the first mass from the origin as θ_1 and the deviation of the second mass from the first mass as θ_2, we can define the coordinate positions of the masses as

$$\begin{cases} x_1 = l_1 \sin(\theta_1) \\ y_1 = -l_1 \cos(\theta_1) \end{cases}$$

$$\begin{cases} x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \\ y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \end{cases}$$

Note that it is easiest if we define the top of the pendulum as zero. The velocities of the masses is then,

$$\begin{cases} \dot{x}_1 = l_1 \dot{\theta}_1 \cos(\theta_1) \\ \dot{y}_1 = l_1 \dot{\theta}_1 \sin(\theta_1) \end{cases}$$

$$\begin{cases} \dot{x}_2 = l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2) \\ \dot{y}_2 = l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2) \end{cases}$$

The Lagrangian,

$$\mathscr{L} = {}^1\!/_2\ m_1(\dot{x}_1^2 + \dot{y}_1^2) + {}^1\!/_2\ m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$

$$\mathscr{L} = {}^{m_1}\!/_2(l_1^2\dot{\theta}_1^2) + {}^{m_2}\!/_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + m_1 g l_1 \cos(\theta_1) + m_2 g(l_1\cos(\theta_1) + l_2\cos(\theta_2))$$

2.9.c Derive the Euler-Lagrange equations

The equations of motion (2.6),

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathscr{L}}{\partial \theta_1} = 0$$

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathscr{L}}{\partial \theta_2} = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

2.10 Spring Pendulum

Consider a stretchable plane pendulum, that is, a mass m suspended from a spring of spring constant k and unstretched length l, constrained to move in a vertical plane. Write down the Lagrangian and obtain the Euler-Lagrange equations.

We can write the coordinates as

$$\begin{cases} x = d\sin(\theta) \\ y = -d\cos(\theta) \end{cases}$$

where we have defined d as the length of the spring and θ as the angular deviation of the pendulum. Unlike other pendulum problems we've looked at, the string length is not constant. The velocities,

$$\begin{cases} \dot{x} = \dot{d}\sin(\theta) + d\dot{\theta}\cos(\theta) \\ \dot{y} = -\dot{d}\cos(\theta) + d\dot{\theta}\sin(\theta) \end{cases}$$

The Lagrangian consists of kinetic energy, spring potential energy, and gravitational potential energy.

$$\mathscr{L} = {}^1\!/\!_2\; m(\dot{d}^2 + d^2\dot{\theta}^2) - {}^1\!/\!_2\; k(d-l)^2 + mgd\cos(\theta)$$

The equations of motion (2.6),

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{d}}\right) - \frac{\partial \mathscr{L}}{\partial d} = 0$$

$$\ddot{d} = d\dot{\theta}^2 - \frac{k}{m}(d-l) + g\cos(\theta)$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = 0$$

$$\ddot{\theta} = -\frac{g}{d}\sin(\theta) - 2\frac{\dot{d}\dot{\theta}}{d}$$

2.11 Bead on a Wire

A wire is bent into the shape given by $y = A|x^n|$, $n \geq 2$ and oriented vertically, opening upward, in a uniform gravitational field g. The wire rotates at a constant angular velocity ω about the y axis, and a bead of mass m is free to slide on it without friction.

2.11.a Find the equilibrium height of the bead on the wire. Consider especially the case $n = 2$

We start by writing the Lagrangian. We can simplify this since we have symmetry about the y-axis, so we end up looking only at $x > 0$,

$$y = Ax^n$$

$$\mathscr{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m}{2}\omega^2 x^2 - mgy$$

$$= \frac{m}{2}(\dot{x}^2 + A^2 n^2 x^{2n-2}\dot{x}^2) + \frac{m}{2}\omega^2\omega^2 x^2 - mgAx^n$$

The equation of motion,

$$(1 + A^2 n^2 x^{2n-2})\ddot{x} + \frac{1}{2}(2n-2)A^2 n^2 x^{2n-3}\dot{x}^2 - \omega^2 x + ngAx^{n-1} = 0$$

In order to find the equilibrium point, we set $\dot{x} = \ddot{x} = 0$,

$$\omega^2 x = ngAx^{n-1}$$

$$x^{n-2} = \frac{\omega^2}{ngA}$$

Converting to height,

$$\left(\frac{y}{A}\right)^{n-2/n} = \frac{\omega^2}{ngA}$$

$$y^{n-2} = \frac{1}{A^2}\left(\frac{\omega^2}{ng}\right)^n$$

Another solution is $y = 0$.

2.11.b Find the frequency of small vibrations about the equilibrium position

To find the frequency of small oscillation, we replace $y = y_0 + \delta$. Going back to the equation of motion, let's define,

$$f(x) = 1 + A^2 n^2 x^{2n-2}$$

$$f(x)\ddot{x} + \frac{1}{2} f'(x)\dot{x}^2 - \omega^2 x + ngAx^{n-1} = 0$$

Looking at the equilibrium point, $x = 0$, for $n > 2$, most terms die,

$$\ddot{\delta} - \omega^2 \delta = 0$$

The frequency of oscillation is the same as the angular velocity of the wire. For $n = 2$,

$$\ddot{\delta} = (\omega^2 - 2gA)\delta$$

We recognize this as the harmonic oscillator,

$$\omega = \sqrt{\omega^2 - 2ga}$$

For the other equilibrium point,

$$x_0 = \left(\frac{\omega^2}{ngA}\right)^{1/(n-2)}$$

The equation of motion becomes

$$f(x_0 + \delta)(\ddot{x} + \ddot{\delta}) + \frac{1}{2} f'(x + \delta)(\dot{x} + \dot{\delta}^2 - \omega^2 (x + \delta) + ngA(x + \delta)^{n-1} = 0$$

Using the binomial expansion and the original equation of motion,

$$f(x_0)\ddot{\delta} - \omega^2 \delta + ngA(n-1)x_0^{n-1}\delta = 0$$

The frequency of oscillation,

$$\omega = \sqrt{\frac{\omega^2 - n(n-1)gAx_0^{n-1}}{f(x_0)}}$$

2.12 Cycloid Equation of Motion

A particle starts at rest and moves along a cycloid whose equation is

$$x = \pm a \cos^{-1}\left(\frac{a-y}{a}\right) + \sqrt{2ay - y^2}$$

There is a gravitational field of strength g in the negative y direction. Obtain and solve the equations of motion. Show that no matter where on the cycloid the particle starts out at time $t = 0$, it will reach the bottom at the same time. [Suggestion: Choose arclength along the cycloid as the generalized coordinate.]

We do as the suggestion and convert to arclength,

$$s = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{2a - y}{\sqrt{2ay - y^2}}$$

$$s = \int \sqrt{1 + \frac{(2a-y)^2}{2ay - y^2}}\, dy = \int \sqrt{\frac{2a}{y}}\, dy$$

$$s = 2\sqrt{2ay}$$

The Lagrangian is given by

$$\mathscr{L} = \frac{1}{2} m\dot{s}^2 - mgy$$

$$\mathscr{L} = \frac{1}{2} m\dot{s}^2 - \frac{mgs^2}{8a}$$

The equation of motion given by this Lagrangian,

$$\ddot{s} + \frac{gs}{4a} = 0$$

We recognize this as a harmonic oscillator,

$$s(t) = A\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{g}{4a}}$$

Furthermore, we know that a pendulum has the same period regardless of starting position.

2.13 Spring

Two masses m_1 and m_2 are connected by a massless spring of spring constant k. The spring is at its equilibrium length and the masses are both at rest; there is no gravitational field. Suddenly m_2 is given a velocity v. Assume that v is so small that the two masses never collide in their subsequent motion. Describe the motion of both masses. What are their maximum and minimum separations?

We start by writing the Lagrangian,

$$\mathscr{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x-l)^2$$

$$x = x_2 - x_1$$

Furthermore, we define the center of mass to be the origin,

$$m_1x_1 + m_2x_2 = 0$$

$$x_1 = -\frac{m_2 x}{m_1 + m_2}$$

$$x_2 = \frac{m_1 x}{m_1 + m_2}$$

Rewriting the Lagrangian in terms of the separation between the two masses,

$$\mathscr{L} = \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}k(x-l)^2$$

The equation of motion,

$$\mu\ddot{x} + k(x-l) = 0$$

$$x(t) = l + \frac{v}{\omega}\sin(\omega t)$$

$$\omega = \sqrt{\frac{k}{\mu}}$$

From this, we can determine the maximum and minimum separation by setting $\sin(\omega t)$ equal to 1 and -1 respectively.

$$x_1(t) = -\frac{m_1}{m_1 + m_2}\left(l + \frac{v}{\omega}\sin(\omega t)\right)$$

$$x_2(t) = \frac{m_1}{m_1 + m_2}\left(l + \frac{v}{\omega}\sin(\omega t)\right)$$